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Optimum Design of Multiple-Constraint-Layered Systems for Vibration Control

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An analytical formulation is presented for predicting the stiffness and damping of constrained layered beams that have multiple viscoelastic damping layers. A model is derived for symmetrical setups using variational methods. The equations to evaluate the stiffness and damping are derived in closed form and can be evaluated for different boundary conditions. The complex modulus approach is used to model the elastic and shear moduli of the viscoelastic material. A parametric analysis has been conducted to study the effects of different parameters on the damping and stiffness of the system under simply supported boundary conditions. The optimization process to obtain structural and material parameters maximizing system damping for a given temperature and frequency range is also presented.

Nomenclature

E	=	Young's modulus of the constraining layer
E_c	=	complex modulus of the viscoelastic material
G	=	shear modulus of the constraining material
G_c	=	complex shear modulus of the viscoelastic material
h	=	layer thickness
L	=	length of the beam
M	=	system mass
Q_{ij}	=	element of the transformed reduced stiffness matrix
u	=	longitudinal displacement of a point in the neutral plane of a layer
w	=	transverse displacement of a point in the neutral plane of a layer
w_i	=	imaginary part of the natural frequency
w_r	=	real part of the natural frequency
Z_k	=	distance from the point in the kth layer to the neutral axis of the layer
ε	=	normal strain
η	=	loss factor of the mulitlayered system
η_c	=	loss factor of the viscoelastic material
ρ	=	material density
ϕ	=	slope of transverse displacement
ψ	=	shear strain in the viscoelastic material
ω	=	frequency of vibration, rad/s

Introduction

IN machines and structures subjected to dynamic loadings, vibrations often induce serious problems leading to material fatigue, noise, and other failures. Therefore, its reduction or elimination is crucial in the design and utilization of these devices. One approach to eliminating or reducing the problems caused by vibrations is the application of passive damping treatment in the structure. Passive damping methods involve the application of soft, viscoelastic materials (VEM). These materials can either be designed into the structure or added later, after the design process is completed. The maximization of the structure's damping is directly connected to the selection of certain design parameters. A system's damping is a function of the steady-state frequency of vibration and the damping parameters within the structure. To predict and prevent vibrations, these parameters have to be calculated, modeled, measured, or in a broader sense, assumed. In the past, the most investigated structure has been a three-layered beam. Usually, this structure consists of a basic layer, a viscoelastic damping layer, and a relatively stiff constraining layer. Owing to its physical appearance, it is often referred to as a "sandwich beam."

Constrained-layer damping treatments are among the most efficient methods of introducing damping into a structure. When the whole multilayered beam is subjected to cyclic bending, the damping layer is primarily subjected to shear strain due to the relatively stiff constraining layer on top and causes friction between the long-chain molecules of the damping material. Therefore energy dissipation occurs in each bending cycle and results in a smaller vibration amplitude. A multiple-layered beam can be assembled by taking more than one damping–constraining layer combination. These multilayered beams are referred to as multiple-constraint-layered systems.

Numerous researchers have successfully implemented the passive constrained layer (PCL) and the active constrained layer (ACL). In 1959, Kerwin[1] and Ross et al.[2] presented a general analysis of a VEM structure. The damping is attributed to extensional and shear deformations of the viscoelastic layers. They applied this theory to a number of practical damping treatments. The experimental data for different treatments confirmed the theory. In the 1960s, researchers extended Kerwin's work on the Ross–Kerwin–Ungar basic assumptions. Diraranto[3] developed sixth-order equations of motion in terms of axial displacements and the closed-form solution. Mead and Markus[4,5] developed sixth-order equations of motion for transverse displacement, which was then applied to certain boundary conditions. The energy method has been applied to many types of problems. Rao[6] presented the equations of motion and boundary conditions using the energy method. He solved the equations numerically and presented a practical design guideline. Similarly to Rao's theory, Cottle[7] used Hamilton's principle to derive equations of motion.

The damping of the treatment could be increased by adding a passive stand-off layer (PSOL) and a slotted stand-off layer (SSOL) to the layered systems. This point was first proposed in 1959. Many researchers have studied these systems in recent years. Falugi[8] and Parin et al.[9] conducted theoretical and experimental work on a

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four-layered panel and a five-layered beam with a PSOL treatment. Rogers and Parin[10] and Yellin et al.[11] have performed experimental investigations and demonstrated that PSOL treatment increased damping significantly in aeronautical structures and beams. Yellin et al.[12] and Yellin and Shen[13] developed normalized equations of motion for beams, fully treated with PSOL using a nonideal-stand-off-layer assumption. The equations were solved using the method of distributed transfer functions.[14] The experimental data confirmed the theory. The intracell boundary conditions[15] were used to construct a transfer matrix of basic unit cells. Singh and Gupta[16] studied the damped–free vibrations of composite cylindrical shells using the first-order shear deformation theories.

The concept of PCL treatment has been applied to engineering design and optimization of many practical dynamic structures. The finite element method has been used for both unconstrained and constrained damping in complex cases to determine optimum locations and thickness of additive damping and sensitivity of damping layers in partial coverage cases.[17,18] A multiparameter optimization study was carried out to achieve a dynamically optimal configuration of viscoelastic layered systems.[19] Lunden[20] proposed that optimum distribution of unconstrained distributed damping on beams and frames reduces the response for the same weight or cost of additive treatment. The objective was to minimize resonant vibration subject to constraints on the weight or cost of the additive damping. The primary optimum objectives of the study done by Lall et al.[21] were loss-factor and displacement-response optimization with design variables such as the layer material densities, thickness, and temperature. For optimization studies of a partially covered plate with constrained VEM, the objective function is to maximize the system loss factor, with design parameters such as position and dimension, while restricting the patch coverage area.[19] The genetic algorithm approach was also used for the optimization design of a cantilever sandwich plate; the objective was to maximize the twist in the desired direction while minimizing the weight of the structure.[22]

The objective of this research was to develop an analytical model to study the vibration and damping of multiple-layered beam systems that have an arbitrary number of viscoelastic layers. The effects of different factors, such as temperature, number of layers, and thickness of different materials were investigated using the developed model and numerical methods. An optimization scheme considering damping, weight, natural frequency, layer number, and thickness is also presented in this paper.

Derivation of the Governing Equations

General Formulation for Layered Systems

For the general case of a layered system, one has to distinguish between systems that have their neutral axis on an elastic layer, and symmetric systems that have their neutral axis on a viscoelastic layer. Hence, the general solution must be stated in two different equations, depending on the layer setup of the system.

Composite System with $2n - 1$ Viscoelastic Layers

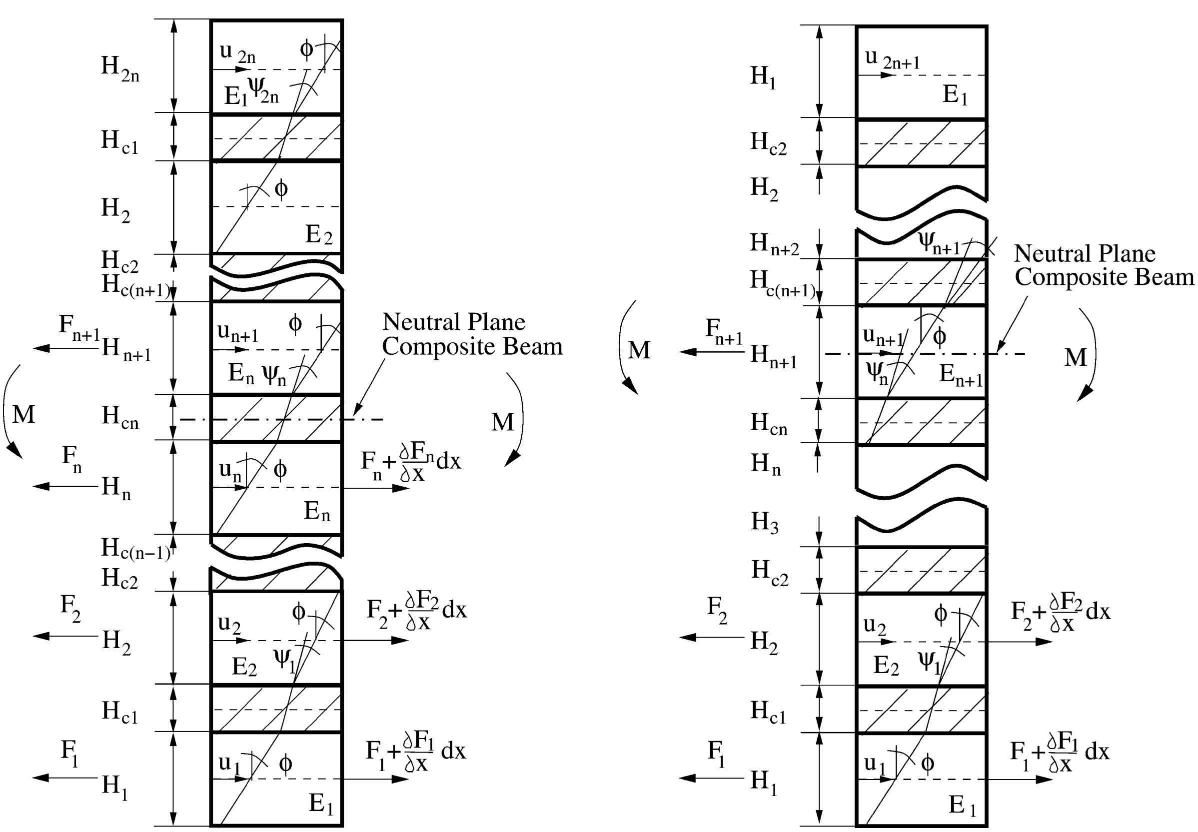
A system with $2n - 1$ viscoelastic layers is considered first. This system has a unit width, a thickness H_n of the nth constrained layer, and an nth viscoelastic layer thickness of H_{cn}. For $2n - 1$ viscoelastic layers, the neutral plane of the composite beam is situated in a viscoelastic layer as shown in Fig. 1a. The beam is considered to be symmetric, implying that the properties of layers having the same distance from the composite beam neutral axis are assumed to be identical ($H_1 = H_{2n}, \ldots,$ and $E_1 = E_{2n}$). The following assumptions are made when the equations are derived:

1) The composite beam has a unit width, $2n - 1$ damping layers, and $2n$ laminates (stiff elastic layers). It is symmetric with respect to the middle plane (xy) of the nth damping layer.

2) It is assumed that normal sections in each layer remain planar and continuous before and after deformations and there is no slip at the interfaces.

3) The deformations account for extension and transverse shear deformations in damping layers and only extension deformations in stiff elastic layers.

4) Transverse displacement is the same for every layer of the composite beam, and only the transverse directional inertia of the



a) b)

Fig. 1 Constrained system setup with a) $2n - 1$ and b) $2n$ viscoelastic layers.

composite beam is dominant, with longitudinal and rotary inertia of the beam negligible.

5) In the current research, it is assumed that the damping occurs mainly through the shear strain in the viscoelastic layer, and the complex modulus approach is used to represent both the extensional and shear moduli of the viscoelastic material. This causes the eigenvalues of the system to be complex and, generally, numerical techniques are needed to obtain the system modal-loss factors and resonance frequencies.

Because the bending of the beam is assumed to be uniform in all constraining layers ($\phi = \text{constant}$), shear angles of viscoelastic layers having the same distance to the composite-beam neutral plane are assumed to be identical; that is,

$$\psi_1 = \psi_{2n-1}, \psi_2 = \psi_{2n-2}, \ldots, \psi_{n-1} = \psi_{n+1}$$

The longitudinal displacement u_k in an elastic layer can be expressed in geometrical terms. The considered kth layer is positioned in the lower half of the symmetric beam. By doing so for any kth layer, it can be found that

$$u_k = u_0 - \frac{H_{cn}}{2}\left(\frac{\partial w}{\partial x} - \psi_n\right) - \frac{\partial w}{\partial x}H_n - \left(\frac{\partial w}{\partial x} - \psi_{n-1}\right)H_{c(n-1)} - \frac{\partial w}{\partial x}H_{n-1} - \cdots - \left(\frac{\partial w}{\partial x} - \psi_k\right)H_{ck} - \frac{\partial w}{\partial x}\frac{H_k}{2} - Z_k\frac{\partial w}{\partial x}$$

$$k = 1, 2, \ldots, n \qquad (1)$$

Here u_0 is the longitudinal displacement of a point in the neutral plane of the beam. A variable D_k is introduced as the distance between the middle plane of the kth constrained layer and the neutral plane of the composite beam:

$$D_k = H_{cn}/2 + H_n + H_{c(n-1)} + H_{n-1} + \cdots + H_{ck} + H_k/2,$$

$$k = 1, \ldots, n-1 \qquad (2)$$

Assuming $u_0 = 0$, using the expression for D_k in Eq. (1), we get

$$u_k = -D_k\frac{\partial w}{\partial x} + \frac{H_{cn}}{2}\psi_n + \sum_{i=k}^{n-1} H_{ci}\psi_i - Z_k\frac{\partial w}{\partial x},$$

$$k = 1, 2, \ldots, n-1 \qquad (3)$$

In a similar way, the longitudinal displacement of the kth viscoelastic layer can be expressed in geometrical terms as

$$u_k^c = u_0 - \frac{H_{cn}}{2}\left(\frac{\partial w}{\partial x} - \psi_n\right) - \frac{\partial w}{\partial x}H_n - \left(\frac{\partial w}{\partial x} - \psi_{n-1}\right)H_{c(n-1)} - \frac{\partial w}{\partial x}H_{n-1} - \cdots - \left(\frac{\partial w}{\partial x} - \psi_k\right)\frac{H_{ck}}{2} - \frac{\partial w}{\partial x}H_{k+1} - Z_k\left(\frac{\partial w}{\partial x} - \psi_k\right) \qquad (4)$$

Here, D_{ck} is the distance between the middle plane of the kth viscoelastic layer and the neutral plane of the composite beam:

$$D_{ck} = \frac{H_{cn}}{2} + \frac{H_{ck}}{2} + \sum_{i=k+1}^{n} H_i + \sum_{i=k+1}^{n-1} H_{ci}$$

$$k = 1, 2, \ldots, n-2 \qquad (5)$$

Using $u_0 = 0$, and using the expression for D_{ck} in Eq. (4), we get

$$u_k^c = -D_{ck}\frac{\partial w}{\partial x} + \frac{H_{cn}}{2}\psi_n + \sum_{i=k+1}^{n-1} H_{ci}\psi_i + \frac{H_{ck}}{2}\psi_k - Z_k\left(\frac{\partial w}{\partial x} - \psi_k\right), \qquad k = 1, 2, \ldots, n-2 \qquad (6)$$

Through the given displacement description, the longitudinal strain of the composite beam ε_k can be determined. For the kth elastic layer, it can be expressed as

$$\varepsilon_k = -D_k\frac{\partial^2 w}{\partial x^2} + \frac{H_{cn}}{2}\psi_n' + \sum_{i=k}^{n-1} H_{ci}\psi_i' - Z_k\frac{\partial^2 w}{\partial x^2}$$

$$k = 1, 2, \ldots, n-1 \qquad (7)$$

The strain of the kth viscoelastic layer can be shown as

$$\varepsilon_{kc} = -D_{ck}\frac{\partial^2 w}{\partial x^2} + \frac{H_{cn}}{2}\psi_n' + \sum_{i=k+1}^{n-1} H_{ci}\psi_i' + \frac{H_{ck}}{2}\psi_k' - Z_k\left(\frac{\partial^2 w}{\partial x^2} - \psi_k'\right), \qquad k = 1, 2, \ldots, n-2 \qquad (8)$$

Since $\psi_{zx}^k = \varepsilon_z^k = 0$, applying the stress–strain relation gives

$$\begin{pmatrix}\sigma_x^k \\ \sigma_z^k \\ \tau_{xz}^k\end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{55}\end{bmatrix}\begin{pmatrix}\varepsilon_x^k \\ \varepsilon_z^k \\ \psi_{xz}^k\end{pmatrix} \qquad (9)$$

where Q_{ij}^k is the normal or shear modulus of elasticity.

The stress–strain relation can be reduced and the strain energy density of the kth layer is expressed by

$$U_o^k = \tfrac{1}{2}E\varepsilon_k^2 \qquad (10)$$

The total strain energy of the elastic layers is given by

$$U_e = 2\sum_{k=1}^{n} U_k = 2\sum_{k=1}^{n}\int_0^L \int_{-(H_k/2)}^{H_k/2} U_o^k \,\mathrm{d}z\,\mathrm{d}x \qquad (11)$$

Applying the stress–strain relation for the viscoelastic layer gives

$$\begin{pmatrix}\sigma_{xc}^k \\ \sigma_{yc}^k \\ \sigma_{zc}^k \\ \tau_{yzc}^k \\ \tau_{xzc}^k \\ \tau_{xyc}^k\end{pmatrix} = \begin{bmatrix} Q_{11c} & Q_{12c} & Q_{13c} & 0 & 0 & 0 \\ Q_{12c} & Q_{22c} & Q_{23c} & 0 & 0 & 0 \\ Q_{13c} & Q_{23c} & Q_{33c} & 0 & 0 & 0 \\ 0 & 0 & 0 & Q_{44c} & 0 & 0 \\ 0 & 0 & 0 & 0 & Q_{55c} & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_{66c}\end{bmatrix}\begin{pmatrix}\varepsilon_{xc}^k \\ \varepsilon_{yc}^k \\ \varepsilon_{zc}^k \\ \psi_{yzc}^k \\ \psi_{xzc}^k \\ \psi_{xyc}^k\end{pmatrix} \qquad (12)$$

where the Q_{ijc}^k is the normal or shear modulus of the viscoelastic material.

Since $\sigma_{yc}^k = \sigma_{zc}^k = \tau_{yzc}^k = \tau_{xyc}^k = 0$, the stress–strain relation can be reduced to

$$\begin{pmatrix}\sigma_{xc}^k \\ \tau_{xzc}^k\end{pmatrix} = \begin{bmatrix} Q_{11c} & 0 \\ 0 & Q_{55c}\end{bmatrix}\begin{pmatrix}\varepsilon_{xc}^k \\ \psi_{xzc}^k\end{pmatrix} \qquad (13)$$

The strain energy density of the kth viscoelastic layer is given by

$$U_{oc}^k = \tfrac{1}{2}E_c\varepsilon_{ck}^2 + \tfrac{1}{2}G_c\psi_k^2 \qquad (14)$$

Thus, the strain energy of the kth damping layer can be integrated by

$$U_c^k = \int_0^L \int_{-(H_{ck}/2)}^{H_{ck}/2} U_{oc}^k \,\mathrm{d}z\,\mathrm{d}x \qquad (15)$$

The total strain energy contributed by viscoelastic layers is given by

$$U_c = U_c^n + 2\sum_{k=1}^{n-1} U_c^k \qquad (16)$$

The kinetic energy can be written in terms of the transverse deflections as

$$T = \frac{1}{2}\int_0^L M\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{17}$$

where

$$M = \rho_c^n H_c^n + 2\rho^n H^n + 2\sum_{k=1}^{n-1}\left(\rho^k H^k + \rho_c^k H_c^k\right)$$

where ρ is the density of different materials. The work done by the external distribution load $q(x, t)$ can be written as

$$W = \int_0^L q(x, t) w(x, t)\, \mathrm{d}x \tag{18}$$

The total strain energy of the system can be expressed by

$$U = U_e + U_c \tag{19}$$

Thus, the system differential equations of motion can be derived by the application of Hamilton's principle

$$\delta \int_{t_1}^{t_2} (T - U + W)\, \mathrm{d}t = 0 \tag{20}$$

Let

$$\int_0^L F\, \mathrm{d}x = T - U + W = T - (U_e + U_c) + \int_0^L q w\, \mathrm{d}x \tag{21}$$

Hamilton's principle gives

$$\delta \int_{t_1}^{t_2} (T - U + W)\, \mathrm{d}t = \delta \int_{t_1}^{t_2}\int_0^L F\, \mathrm{d}x\, \mathrm{d}t \tag{22}$$

By the principles of the calculus of variations, the equations of motion are given by

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w''}\right) - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{w}}\right) + \frac{\partial F}{\partial w} = 0$$

$$\frac{\partial F}{\partial \psi_k} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \psi_k'}\right) = 0 \tag{23}$$

where

$$k = 1, 2, \ldots, n$$

The boundary conditions can also be obtained as

$$\frac{\partial F}{\partial w''}\delta w'\Big|_0^L = 0, \qquad \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w''}\right)\delta w\Big|_0^L = 0, \qquad \frac{\partial F}{\partial \psi_k'}\delta \psi_k\Big|_0^L = 0 \tag{24}$$

After rearrangement of formulations, we get

$$\frac{\partial}{\partial x^2}\left(\frac{\partial F}{\partial w''}\right), \qquad -\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{w}}\right), \qquad \frac{\partial F}{\partial w}$$

$$\frac{\partial F}{\partial \psi_k}, \qquad -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \psi_k'}\right) \tag{25}$$

Using these in Eq. (23), the governing equations of motion of the system can be obtained as

$$D_x w'''' + \sum_{i=1}^{n} DD_i \psi_i''' + M\ddot{w} = 0$$

$$Dx_j w''' + \sum_{i=1}^{n} D_{ji}\psi_i'' + D_{yj}\psi_j = 0 \tag{26}$$

where

$$j = 1, 2, \ldots, n$$

The variables D_x, DD_i, D_{xj}, D_{ji}, and D_{yj} are the coefficients of the equations.

Composite System with 2*n* Viscoelastic Layers

For the case of a composite system with $2n$ viscoelastic layers, the neutral plane of the composite beam will be in a constrained layer, as shown in Fig. 1b. The system is again considered to have unit width with thickness H_n for the nth constraint layer and H_{cn} for the nth viscoelastic layer. The system is assumed to be symmetric; therefore the properties are assumed to be identical for layers having the same distance to the composite system's neutral plane. The shear angles are once more assumed to be identical for layers having the same distance to the neutral plane. Therefore,

$$\psi_1 = \psi_{2n}, \psi_2 = \psi_{2n-1}, \ldots, \psi_n = \psi_{n+1}$$

Evaluating the longitudinal displacement of the kth elastic layer situated in the lower half of the system yields

$$u_k = u_0 - \frac{H_{n+1}}{2}\left(\frac{\partial w}{\partial x}\right) - \left(\frac{\partial w}{\partial x} - \psi_n\right)H_{cn} - \left(\frac{\partial w}{\partial x}\right)H_n - \cdots$$
$$- \left(\frac{\partial w}{\partial x} - \psi_k\right)H_{ck} - \frac{\partial w}{\partial x}\frac{H_k}{2} - Z_k\frac{\partial w}{\partial x}, \qquad k = 1, 2, \ldots, n \tag{27}$$

According to Eq. (27), the distance D_k between the middle plane of the kth constraining layer and the neutral plane of the system is found to be

$$D_k = \frac{H_{n+1}}{2} + \sum_{i=k}^{n} H_{ci} + \sum_{i=k+1}^{n} H_i, \qquad k = 1, 2, \ldots, n-1 \tag{28}$$

Noting that $u_0 = 0$ and substituting the variable D_k into the longitudinal displacement equation, one finds

$$u_k = -D_k\frac{\partial w}{\partial x} + \sum_{i=k}^{n}(H_{ci}\psi_i) - Z_k\frac{\partial w}{\partial x} \tag{29}$$

D_{ck} is the distance between the middle plane of the kth viscoelastic layer and the neutral plane of the composite beam,

$$D_{ck} = \frac{H_{n+1}}{2} + \frac{H_{ck}}{2} + \sum_{i=k+1}^{n}(H_i + H_{ci}) \tag{30}$$

The longitudinal displacement of the kth viscoelastic layer can be found to be

$$u_k^c = -D_{ck}\frac{\partial w}{\partial x} + \sum_{i=k+1}^{n}(H_{ci}\psi_i) + \frac{H_{ck}}{2}\psi_k - Z_k\left(\frac{\partial w}{\partial x} - \psi_k\right)$$
$$k = 1, 2, \ldots, n-1 \tag{31}$$

As before, when the stress–strain relation is applied, the total strain energy of the elastic layers is given by

$$U_e = 2\sum_{k=1}^{n} U_k + U_{n+1} \tag{32}$$

By applying the stress–strain relation to the viscoelastic layers, the total strain energy of the viscoelastic layers is found to be

$$U_c = 2\sum_{k=1}^{n} U_c^k \tag{33}$$

The kinetic energy and the work done by the external distribution load have the same expressions as in the previous $2n-1$ layer case. Applying Hamilton's principle yields equations of motion similar to those in Eq. (26).

Solution Scheme

To find the loss factors and resonance frequencies of the multilayered system, a MATLAB program was developed that evaluates the complex stiffness of the considered system using the equations developed here. For the case of a simply supported composite beam under harmonic vibration, the comparison functions can be given by

$$W = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi x}{L}\right) e^{iwt} \tag{34}$$

$$\psi_i = \sum_{k=1}^{\infty} B_{ik} \cos\left(\frac{k\pi x}{L}\right) e^{iwt} \tag{35}$$

Considering the first m modes of vibration, using Eqs. (34) and (35) in Eq. (26), the following equations can be obtained after rearrangement:

$$\begin{bmatrix} D_x\left(\frac{m\pi}{L}\right)^4 - Mw^2 & DD_1\left(\frac{m\pi}{L}\right)^3 & \cdots & DD_i\left(\frac{m\pi}{L}\right)^3 & \cdots & DD_n\left(\frac{m\pi}{L}\right)^3 \\ D_{x1}\left(\frac{m\pi}{L}\right)^3 & D_{11}\left(\frac{m\pi}{L}\right)^2 - D_{y1} & \cdots & DD_{1i}\left(\frac{m\pi}{L}\right)^2 & \cdots & DD_{1n}\left(\frac{m\pi}{L}\right)^2 \\ D_{x2}\left(\frac{m\pi}{L}\right)^3 & D_{21}\left(\frac{m\pi}{L}\right)^2 & \cdots & & & \\ \vdots & & & & & \\ D_{xi}\left(\frac{m\pi}{L}\right)^3 & D_{i1}\left(\frac{m\pi}{L}\right)^2 & \cdots & DD_{ii}\left(\frac{m\pi}{L}\right)^2 - D_{yi} & \cdots & DD_{in}\left(\frac{m\pi}{L}\right)^2 \\ \vdots & & & & & \\ D_{xn}\left(\frac{m\pi}{L}\right)^3 & D_{n1}\left(\frac{m\pi}{L}\right)^2 & \cdots & DD_{ni}\left(\frac{m\pi}{L}\right)^2 & \cdots & DD_{nn}\left(\frac{m\pi}{L}\right)^2 - D_{yn} \end{bmatrix} \begin{pmatrix} A_m \\ B_{1m} \\ B_{2m} \\ \vdots \\ B_{nm} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Rewriting, we get

$$\begin{bmatrix} k_{11} - Mw^2 & k_{12} & \cdots & k_{1i} & \cdots & k_{1n} \\ k_{12} & k_{22} & \cdots & k_{2i} & \cdots & k_{2n} \\ k_{13} & k_{23} & \cdots & & & \\ \vdots & & & & & \\ k_{1i} & k_{2i} & \cdots & k_{ii} & \cdots & k_{in} \\ \vdots & & & & & \\ k_{1n} & k_{2n} & \cdots & k_{in} & \cdots & k_{nn} \end{bmatrix} \begin{pmatrix} A_m \\ B_{1m} \\ B_{2m} \\ \vdots \\ B_{nm} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{36}$$

The determinant of the coefficient matrix should be zero for a nontrivial solution:

$$\begin{vmatrix} k_{11} - Mw^2 & k_{12} & \cdots & k_{1i} & \cdots & k_{1n} \\ k_{12} & k_{22} & \cdots & k_{2i} & \cdots & k_{2n} \\ k_{13} & k_{23} & \cdots & & & \\ \vdots & & & & & \\ k_{1i} & k_{2i} & \cdots & k_{ii} & \cdots & k_{in} \\ \vdots & & & & & \\ k_{1n} & k_{2n} & \cdots & k_{in} & \cdots & k_{nn} \end{vmatrix} = 0 \tag{37}$$

This is rewritten as

$$k_{11} + \frac{\sum_{j=2}^{n+1} k_{1j}\,\Delta k_{1j}}{\Delta k_{11}} - w^2 M = 0 \tag{38}$$

The study of the system resonance frequencies ω and modal loss factors η are our primary interest here. Hence, it is appropriate to assume that the layered beam system is under harmonic vibration with the complex resonance frequency $\omega^2 = \omega_r^2(1 + i\eta)$. It has been shown in Ref. 4 that a special class of complex modes exist when the system is externally excited by specific "damped loads," which are also complex and which are proportional to the local inertia loadings of the mode. When the equation is solved, the damped complex resonance frequency is found to be

$$\omega^2 = k_{11} + \frac{\sum_{j=2}^{n+1} k_{1j}\,\Delta k_{1j}}{\Delta k_{11}} \Bigg/ M \tag{39}$$

$$w_r^2 = \mathrm{Re}(\omega^2), \qquad w_I^2 = \mathrm{Im}(\omega^2) \tag{40}$$

Hence, the resonance frequency w_r and loss factor η can be calculated for each mode:

$$w_r = \sqrt{\mathrm{Re}(\omega^2)} \tag{41}$$

$$\eta = \frac{\mathrm{Im}(\omega^2)}{\mathrm{Re}(\omega^2)} \tag{42}$$

The loss factor is evaluated as the ratio of the imaginary to the real part of the square of the system's complex natural frequency.

The material properties of the damping layer and their frequency and temperature dependence are needed as inputs to the program.

Considerable research is being carried out in the area of analytical modeling of frequency and temperature effects on the damping and modulus of viscoelastic materials. Although data on the shear modulus and loss factor of damping materials corresponding to a frequency and temperature are available in the data sheets supplied by the manufacturers, these data are not useful here because we do not know the resonant frequencies of the system. However, based on the experimental data and curve-fitting, some formulas are available to find the values of E_c, G_c, and η_c corresponding to a frequency and a temperature. The empirical formulas given in He and Rao's paper[23] are used in this study.

In the numerical approach, however, the frequency is needed at the very beginning to evaluate the properties of the viscoelastic material because these properties are frequency- and temperature-dependent. Therefore, an iterative process needs to be developed. To do so, an approximate natural frequency is first determined for a reference beam having the same overall geometrical setup as the considered multilayered system and having the same Young's modulus as the first elastic layer. Assuming boundary conditions, length, and mode number, the approximate natural frequency can be determined. This determined natural frequency is then put back to calculate the exact natural frequency of the multilayer system. Since the frequencies should be identical, the initial value is compared to the new value. This process is repeated until the difference between the two values is smaller than 5%. This process needs to be repeated in the desired temperature range a number of times to obtain the damping and natural frequencies within that range.

The solution scheme used here is the classical Ritz method. It is fairly easy for the simply supported boundary conditions, but it is rather lengthy and tedious for other boundary conditions. The accuracy and convergence of the solution depend on the chosen functions. The Lagrange multiplier method can be used to solve the problem for any other boundary conditions. The assumed deflection function need not satisfy all the boundary conditions. Those unsatisfied boundary conditions can be imposed as constraints using Lagrange multipliers. Then, some simple deflection functions can be used to make the expression and solution simpler.[24]

Comparison with Mead's Equation

For a three-layer system, the equations of motion become

$$D_x w^{(4)} + DD_1 \psi_1^{(3)} + M\ddot{w} = 0 \tag{43}$$

$$Dx_1 w^{(3)} + D_{11}\psi_1^{(2)} + D_{y1}\psi_1 = 0 \tag{44}$$

The variables D_x, DD_1, D_{x1}, D_{11}, and D_{y1} are the coefficients of the equations for a three-layer system.

Let

$$k = (D_{11}D_x - DD_1 Dx_1)/D_{y1} \tag{45}$$

From Eqs. (43) and (44), after the rearrangement of formulations, we get

$$w^{(6)} + \frac{D_x}{k} w^{(4)} + \frac{D_{11}M}{D_{y1}k}\ddot{w}^{(2)} + \frac{M}{k}\ddot{w} = 0 \tag{46}$$

When the effect of extension deformations in damping layers is neglected, the assumption of this will be exactly the same as in Mead's work (see Ref. 4). The equation becomes

$$w^{(6)} + \frac{2G_c\left[h^2 + 3(h_c + h)^2\right]}{Eh_c h^3} w^{(4)} + \frac{6M}{Eh^3}\ddot{w}^{(2)} + \frac{12G_c M}{E^2 h_c h^4}\ddot{w} = 0 \tag{47}$$

If the applied load term is not considered, Eq. (47) is exactly the same as Mead's equation.

Optimum Structure Design

An optimum design analysis is carried out for a multilayer system. Two parts are considered in this multiobjective optimization problem. The first objective is to maximize the modal loss factor; the second is the minimization of the total mass of the system. Both of the objectives are subjected to constraints on design variables and other requirements such as mass, thickness, temperature, and frequency range. In this way, it is easy to find the optimum configuration including layer thickness, layer number, and material in a required range. The optimization problem will thus be formulated to minimize the function

$$F(h_i, h_{ci}) = M - C \times \eta \tag{48}$$

where M is the mass of the system, and C is a constant.

The constraints imposed are as follows: $M_l \leq M \leq M_u$; M_l and M_u are the lower and upper bounds, respectively; $f_l \leq f \leq f_u$; Natural frequency should be in the required range. Here f is the modal natural frequency, and f_l and f_u are the lower and upper bounds, respectively; $h_l \leq h_i \leq h_u$; h_i is the thickness of the ith elastic layer, and h_l and h_u are the lower and upper bounds of the thickness, respectively; and $h_{cl} \leq h_c \leq h_{cu}$; h_c is the thickness of the ith viscoelastic layer, and h_{cl} and h_{cu} are the lower and upper bounds of the thickness, respectively.

Numerical Results Analysis

This section presents the results determined by investigating different system setups using the developed MATLAB program. Specific setups are assumed, the program is run, and the results are presented and discussed.

Effects of Multiple Layers

The temperature is the most important single factor influencing the characteristics of the viscoelastic damping material and therefore the performance of the assembled multiple layered system. By examining the natural frequency equation, it can be seen that the temperature affects the Young's modulus and the shear modulus of the viscoelastic layers and therefore the damping of the assembled system. This effect is due to the change in the physical properties of the viscoelastic materials with temperature. Typical variations of the damping and stiffness of viscoelastic materials with temperature can be found in standard textbooks.[25]

The best damping performance is achieved when the damping material operates near its transition region. This is a result of lower forces that cause a rearrangement of physical bonds in the viscoelastic material in each bending cycle. This rearrangement can be interpreted as friction on a molecular level that transforms energy of motion to heat energy.

The damping capacity strongly depends on the thickness of the viscoelastic material within the layered system. Hence, when the number of layers is increased, an increase in the loss factor is expected due to more damping material in the system. The variation of the loss factor and stiffness of different systems with temperature for the first three modes is shown in Fig. 2. Here for each successive case, a new damping and constraining layer was added to the system, leading to analysis for three, five, seven, and finally nine layers. The system had a uniform steel layer thickness of 1 mm and a uniform viscoelastic layer thickness of 0.8 mm, and the viscoelastic damping material chosen was the 3M-ISD110. The beam length was 1 m, and the system was under simply supported boundary conditions. It can be seen that the loss factor and the natural frequency increase and the maximum damping regions shift to the left (with temperature) when the layer number increases. It is also found that the optimum damping region of the system with the 3M-ISD 1101 material is around 50°C.

It can also be observed that the natural frequency is relatively high in the lower temperature region. The natural frequency decreases significantly with increasing temperature. This means that the stiffness decreases significantly as well. For the chosen damping material, the temperature where the viscoelastic material goes from transition to rubbery or even into the flow region is seen to be around

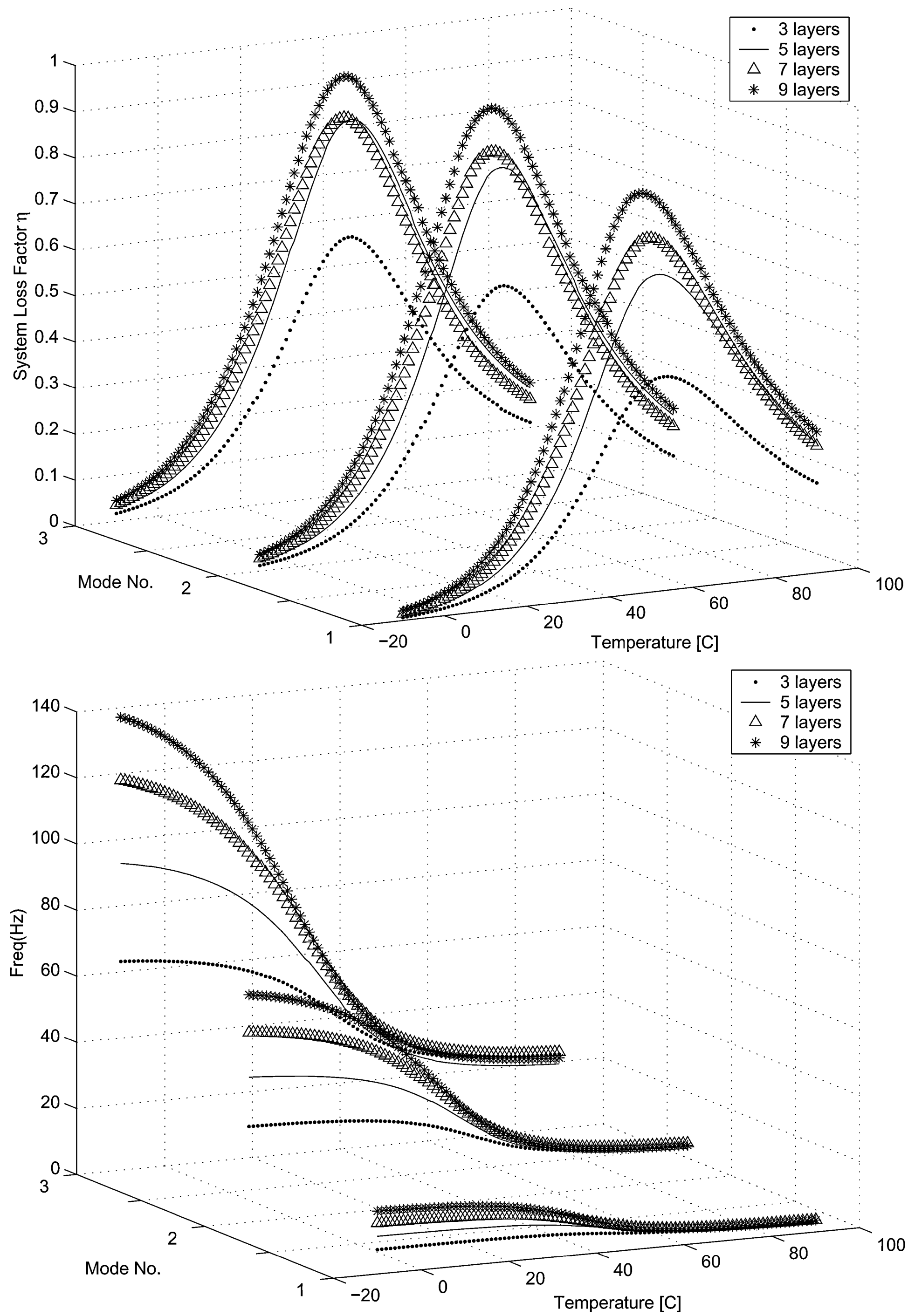


Fig. 2 Loss factor and natural frequency for setups with different layers for different modes (damping material 3M-ISD 110).

80°C. For temperatures greater than this, the damping material tends to become very soft or even starts to undergo an irreversible thermal decomposition process. This leads to lower values of stiffness, which can also be explained at a molecular level through a loosening of the physical bonds between the long-chain molecules.

Effects of Layer Thickness

Once a suitable damping material is selected based on the operation temperature range, the design parameters of the system have to be addressed. The overall thickness of the system and the thickness of both the damping and the elastic (constraining) layers must be selected. Both of these affect the damping and stiffness of the multilayered system. Since here only symmetric systems are considered, a change in thickness always has to occur in corresponding layer pairs.

For the following analysis, the overall thickness of the system was held constant for each test. This might be an important design issue, since in practical applications, a variation of the overall geometrical dimensions is often limited or even impossible due to space, weight, or other criteria. This means that a change of any layer in one direction causes a change of another layer, because the overall thickness remains constant. The boundary conditions were selected to be simply supported and the first mode of vibration was considered.

In Fig. 3, the total system thickness was kept constant at 5 mm. The setup for different configurations (curves 1–4) is shown in Table 1. The normalized system stiffness is the composite system's stiffness, normalized by the stiffness value of a system having the same overall thickness but consisting only of the material of the outermost layers. It can be found that the distance from the thicker layer to the system's

Table 1 Different layer configurations (in millimeters) for the analysis having overall constant thickness for a five-layered system

	Fig. 3			
Layer	Curve 1	Curve 2	Curve 3	Curve 4
Elastic layer 5	1.0	0.8	1.0	1.0
Damping layer 4	1.0	1.2	1.2	0.8
Elastic layer 3	1.0	1.0	0.6	1.4
Damping layer 2	1.0	1.2	1.2	0.8
Elastic layer 1	1.0	0.8	1.0	1.0

neutral axis has some influence on the damping characteristics. Also, the temperature at which the maximum damping occurs changes somewhat in different systems, as seen in Fig. 3.

Thickness of the Viscoelastic Layers

To obtain Fig. 4, an approach was chosen in which the total system thickness was held constant and only the thickness of the viscoelastic layers was varied. The variation of the properties for a system with an overall constant thickness, here 17.5 mm, can be seen. The setup for every test is listed in Table 2. There were nine layers in the system, shown in Fig. 4.

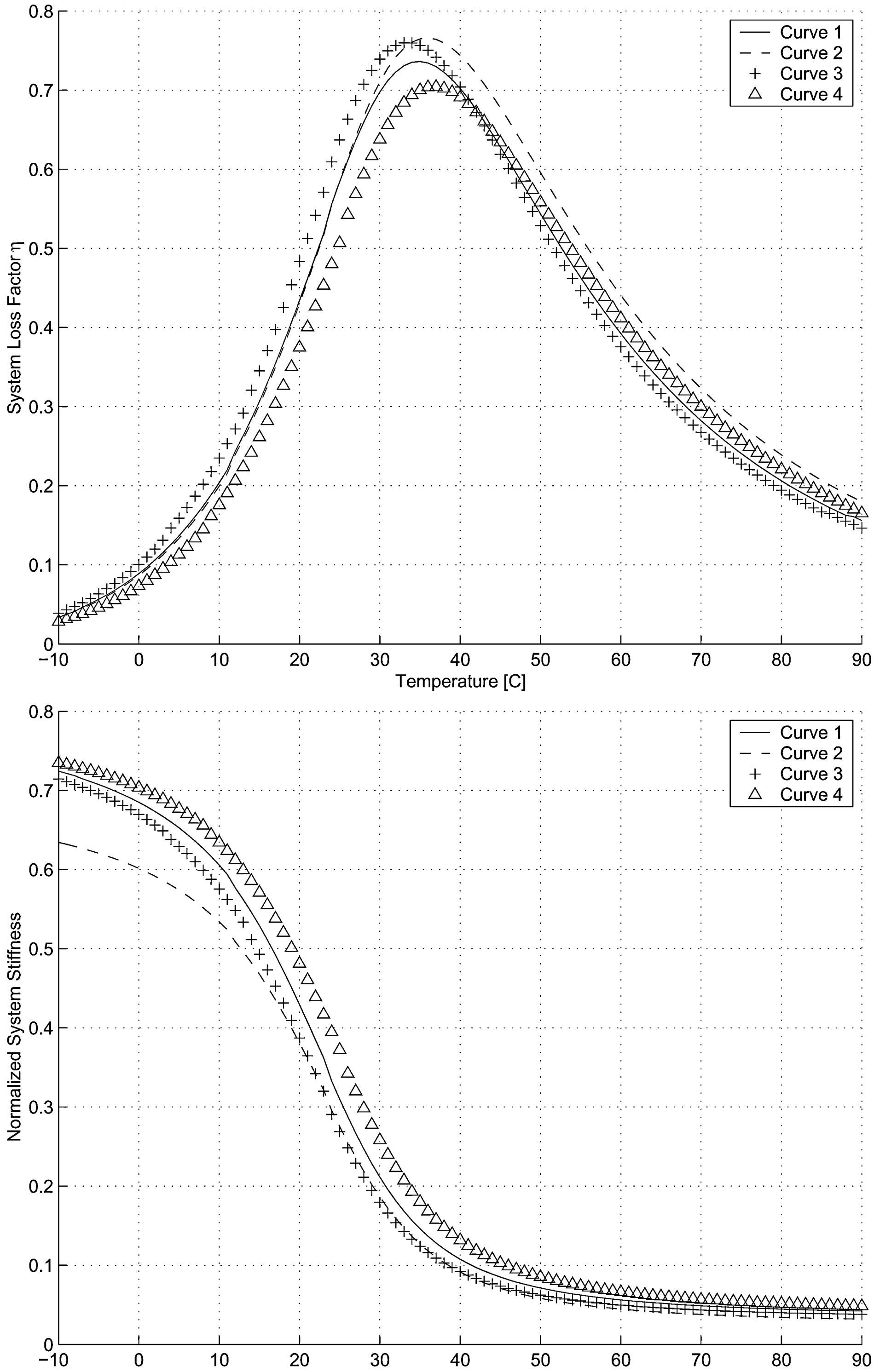


Fig. 3 Loss factor and normalized stiffness for a five-layered system having different configurations (refer to Table 1 for configuration details).

Table 2 Different layer thicknesses (in millimeters) for the analysis having overall constant thickness for nine-layered system

	Fig. 4				Fig. 5			
Layer	Curve 1	Curve 2	Curve 3	Curve 4	Curve 1	Curve 2	Curve 3	Curve 4
Elastic layer 9	2.5	2.5	2.5	2.5	1.0	2.5	0.5	3.0
Damping layer 8	0.8	1.7	0.4	2.1	1.0	1.0	1.0	1.0
Elastic layer 7	2.5	2.5	2.5	2.5	2.5	1.0	3.0	0.5
Damping layer 6	1.7	0.8	2.1	0.4	1.0	1.0	1.0	1.0
Elastic layer 5	2.5	2.5	2.5	2.5	1.0	1.0	1.0	1.0
Damping layer 4	1.7	0.8	2.1	0.4	1.0	1.0	1.0	1.0
Elastic layer 3	2.5	2.5	2.5	2.5	2.5	1.0	3.0	0.5
Damping layer 2	0.8	1.7	0.4	2.1	1.0	1.0	1.0	1.0
Elastic layer 1	2.5	2.5	2.5	2.5	1.0	2.5	0.5	3.0

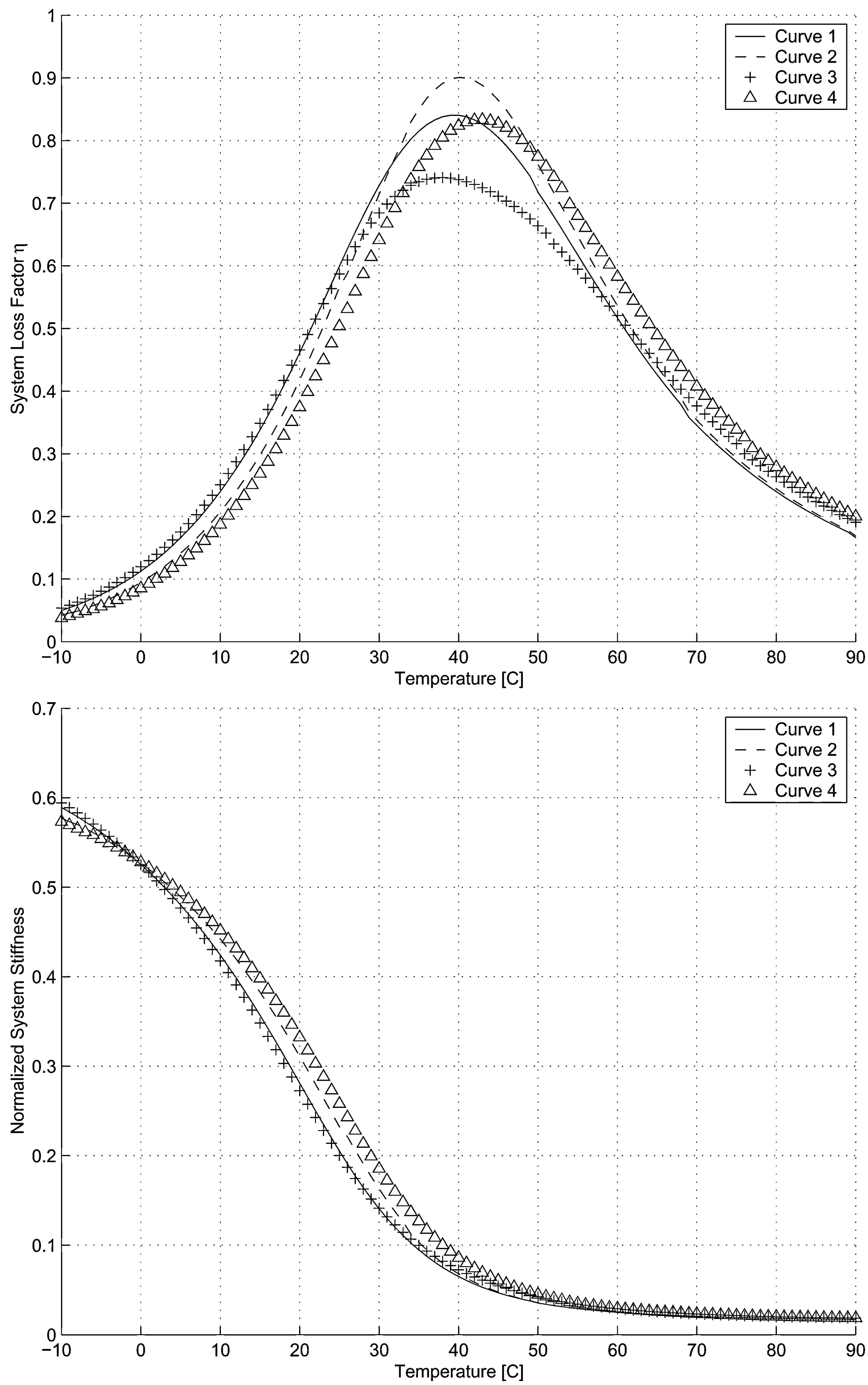


Fig. 4 Loss factor and normalized stiffness for a nine-layered system having different configurations with constant elastic layer thickness (refer to Table 2 for configuration details).

To compare the effects of the layer position, the test was run for a system with two pairs of damping layers, a relatively thick one and a thinner one. Because in multiple-layered systems the position of layers can be changed, a variation in the position of the thicker damping layers was also done, showing the influence of the distance from the thicker layer to the system's neutral axis on the damping characteristics.

Comparing curves 1 and 3 or 2 and 4 in Fig. 4, it can be observed that the maximum damping decreases for the system with a thicker damping layer. Another observation is that the temperature at which the maximum damping happens changes slightly for different setups. This is due to the change in the natural frequency of the system, caused by the different viscoelastic layers in the structure.

The maximum loss factor η for a system having constant thickness also depends on the position of each damping-layer pair. Taking curves 1 and 2 or 3 and 4, it can be seen that the maximum damping increases when the thicker layer is put farther away from the system's neutral axis. Therefore damping of a specific setup can be influenced not only by the layer thickness but also by the way of assembling the different layers within the structure.

When the systems stiffness in Fig. 4 are compared, an almost identical characteristic can be seen for all four setups. This suggests that for systems with constant overall thickness, varying the damping layer thickness does not have a significant effect on the system stiffness as long as the total amount of damping material is held constant.

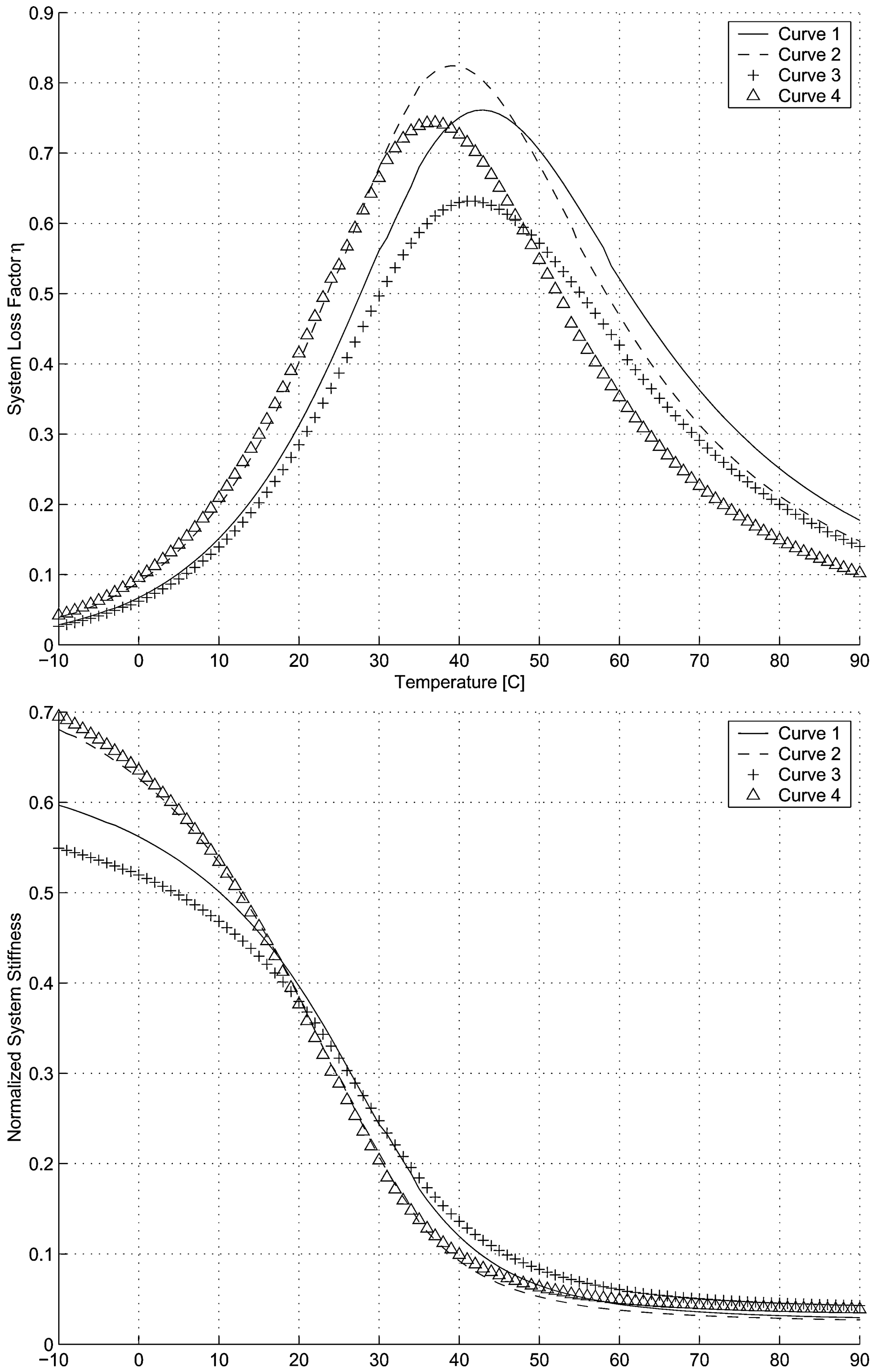


Fig. 5 Loss factor and normalized stiffness for a nine-layered system having different configurations with constant viscoelastic layer thickness (refer to Table 2 for configuration details).

Thickness of the Elastic Layers

Just as the thickness of the arramged layers can be varied, so can the thickness of the elastic constraining layers. Again, taking a system that has a constant overall thickness (12 mm) with setups, as shown in Table 2, and evaluating its properties, Fig. 5 was obtained. One can see that for a variation of the thickness of constraining layers the influence on damping as well as on stiffness is greater here than in the previous case, in which the thickness of the damping layers was varied. From curves 1 and 3 or 2 and 4, it can be concluded that the damping decreases if the elastic layer increases in thickness.

When the thicker elastic layer is arranged on the outside of the system, an increase in loss factor in comparison to the loss factor of the otherwise identical system can be observed (curves 1 and 2 or 3 and 4). The temperature for the maximum loss factor due to a changed natural frequency, as in the viscoelastic layer variation case, shifts toward lower temperatures. However, this shift is more distinctive for the variation of the elastic layer than for the viscoelastic layer.

The relative stiffness of the system changes for variation of the elastic layers more significantly than in the case of a damping-layer variation. Its value can be increased by putting the thicker constraining layer on the outside of the system.

Influence of Different Viscoelastic Materials

A true advantage of multiple-layered systems is the fact that different viscoelastic materials can be employed in different layers of the system to obtain a considerably higher loss factor over a wider temperature range. Doing a comparison analysis for a system having

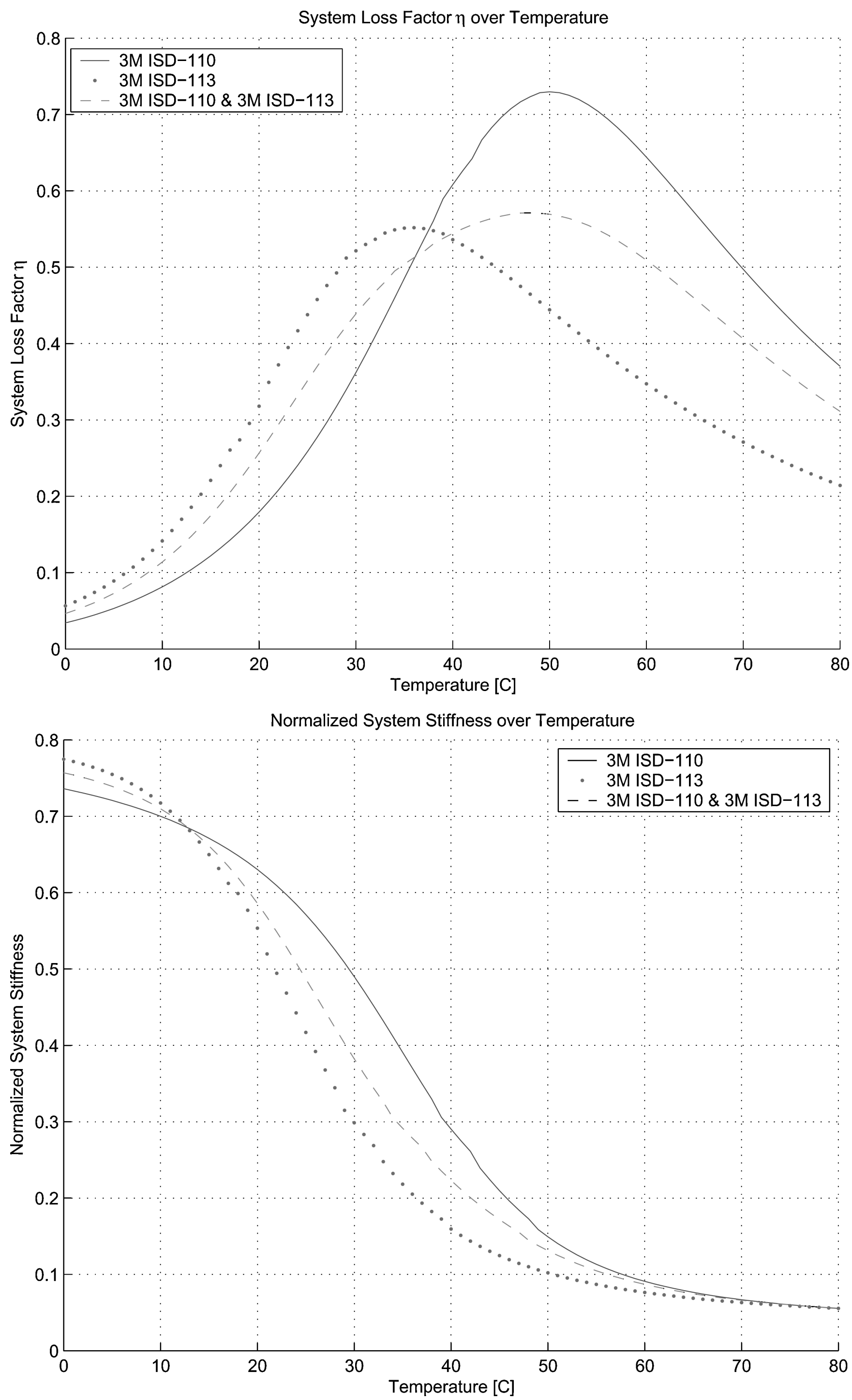


Fig. 6 Loss factor and normalized stiffness for a system with different damping-layer material setups.

seven layers with either one viscoelastic material (3M-ISD 110 or 3M-ISD 113) or two damping materials (3M-ISD 113 and 3M-ISD 110) gives results for the damping characteristics as shown in Fig. 6. The outer viscoelastic layers had thickness of 0.6 mm, the inner one 0.8 mm in order to compensate for the fact that only one layer of this material was present. All elastic layers were set at 1 mm.

The system with 3M-ISD 110 and 3M-ISD 113 materials in one setup results in considerable damping characteristics over the temperature range from 10 to 70°C due to different transition region temperatures for each of the viscoelastic materials. The temperature range in which the damping layers are able to introduce a considerable amount of damping to the system is significantly expanded in comparison to a system with only one damping material in its layers. Therefore the system is able to serve over a wider temperature range, which significantly increases the number of applications.

However, as can be seen in Fig. 6, the loss factor value is not as high as if only one material is used for the damping layers. In setting up a system with two or more different viscoelastic materials, one therefore must match up the different characteristics (peak properties) in order to get an overall system that has the desired property. The change in stiffness over temperature is shown in Fig. 6. It shows that the maximum stiffness value is almost unchanged.

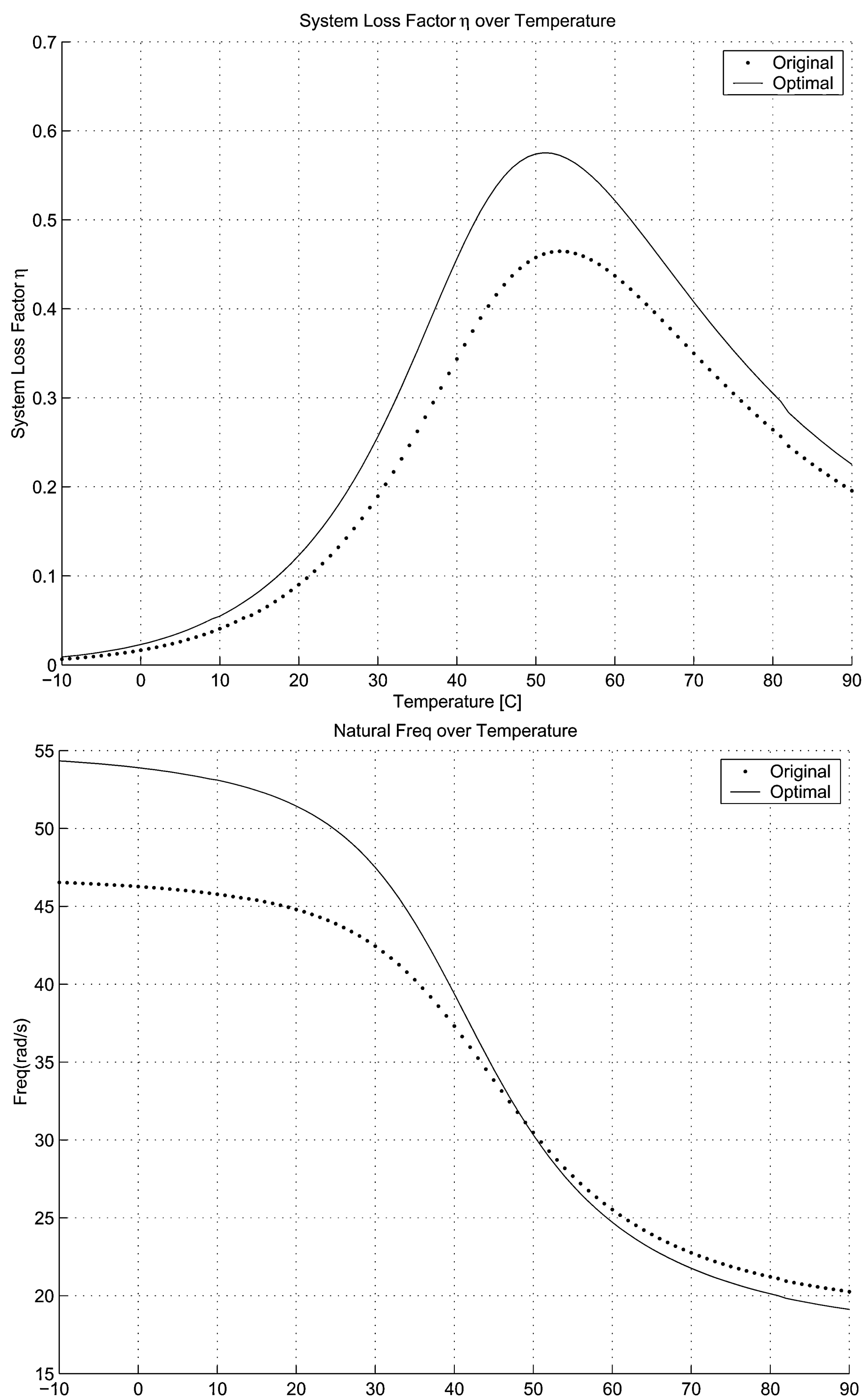


Fig. 7 Optimization of a three-layered system.

Results of Optimization

Simply supported beams with three and seven layers were considered as illustrative examples to implement the optimization procedure outlined before. The damping material used in the system was 3M-ISD 110 and the reference temperature was 45°C. The Young's modulus and density of the elastic layer material were 210 GN/m^2 and 7800 kg/m^3. In the initial design, the thickness of the elastic layer (h_i) and the viscoelastic layer (h_{ci}) were 1.0 and 0.8 mm, respectively.

In the three-layer case, the initial natural frequency of the system was set to 33.8 rad/s. The loss factor η was 0.4157, and the mass was 1.652 kg. The optimization results gave the optimized thickness, $h_1 = 0.9$ mm, $h_{c1} = 1.3$ mm. Using the optimized thickness, the natural frequency and loss factor became 35 rad/s and 0.5485. The mass changed to 1.6 kg in the final design. The results indicate that the loss factor increases significantly and the natural frequency stays in a similar range to the original. Also, the structure becomes lighter than in the original design. The variation of the loss factor and frequency of the systems over temperature are shown in Fig. 7.

In the seven-layer case, the initial natural frequency of the system was set to 47.95 rad/s and the loss factor to 0.7421, mass M was 3.38 kg. After the optimization, the configurations were as follows: $h_1 = 1.1$ mm, $h_2 = 0.2$ mm, $h_{c1} = 1.3$ mm, $h_{c2} = 1.0$ mm, $f = 43$ rad/s, loss factor $= 0.8334$, and $M = 2.4$ kg. The variation of η and f can be seen in Fig. 8.

The advantages of the optimization procedure can be observed from the two example problems: The final design system loss factor is increased significantly and the final structural weight is reduced while simultaneously meeting all performance bounds. However, the optimization will not significantly change the first mode natural frequency of the structure; this is very important in the structure design.

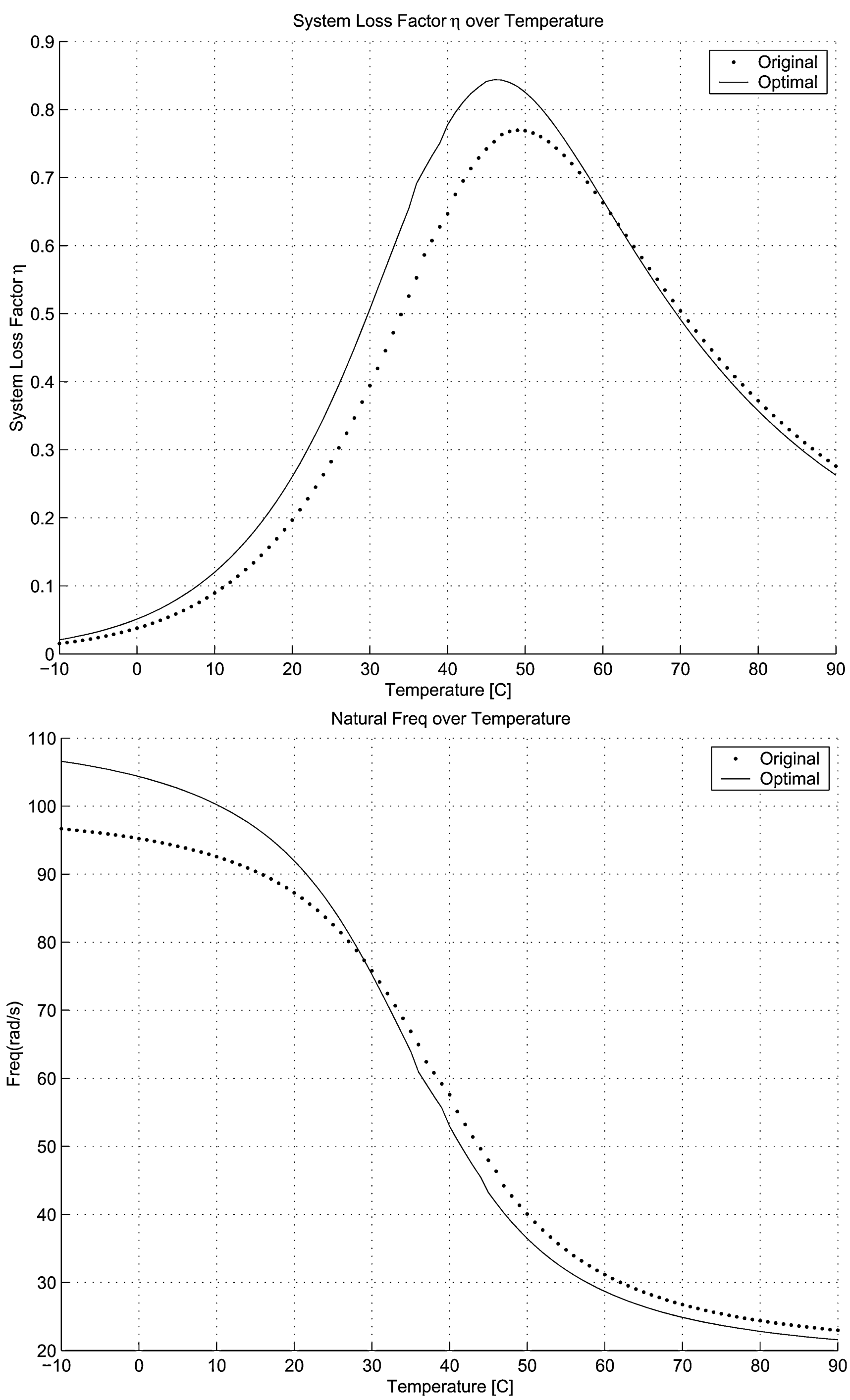


Fig. 8 Optimization of a seven-layered system.

Conclusions

In this research, a comprehensive analytical formulation to study the vibration and damping of a multilayer beam system with viscoelastic damping layers was developed. The vibration characteristics and the effects of damping layers were calculated using a numerical method. A procedure for carrying out an optimum design study has been outlined. Maximizing the system loss factor and minimizing the system weight are two main objectives that were combined into objective functions using the weighting method. Constraints were imposed to make the system satisfy the configuration and natural frequency requirements.

In numerical investigations, the change in damping and the decrease in system stiffness over temperature show that the influence of the ambient conditions, mainly the operating temperature, is significant for a system's damping and stiffness performance. The thickness of the layers has an influence on the performance of the system. A thicker damping layer is generally able to introduce more damping to the system. However, it also results in a loss of stiffness. When a multiple-layered system is assembled, the order in which the different layers are assembled within the system can be selected and this affects the properties of the system. A change in position, however, does not cause significant changes, unlike a change in layer thickness. It can therefore be stated that the number of layers, the thickness of each layer, and the position in the system are all factors that have an influence on the damping properties of a system and can therefore be varied to obtain an assembled system setup that is able to achieve the desired performance under the specified conditions. It is shown through the example problems that a significant increase in loss factor and reduction in weight of the systems can be obtained by the use of the optimization scheme proposed in this paper.

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